(Thesis Synopsis - 5000 words)

STUDY OF QUANTUM RESOURCES TO DEVISE ADVANCED COMMUNICATION PROTOCOLS

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Submitted by

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List of Publications

*1. *Classical analogue of quantum superdense coding and communication advantage of a single quantum system*;

Ram Krishna Patra, Sahil Gopalkrishna Naik, Edwin Peter Lobo, Samrat Sen, Tamal Guha, Some Sankar Bhattacharya, Mir Alimuddin, and Manik Banik;

Quantum **8**, 1315 (2024). [**RKP 1**]

*2. *Principle of Information Causality Rationalizes Quantum Composition;*

Ram Krishna Patra, Sahil Gopalkrishna Naik, Edwin Peter Lobo, Samrat Sen, Govind Lal Sidhardh, Mir Alimuddin, and Manik Banik;

Phys. Rev. Lett. **130**, 110202 (2023). [**RKP 2**]

*3. *Certifying beyond quantumness of locally quantum no-signaling theories through a quantum-input Bell test;*

Edwin Peter Lobo, Sahil Gopalkrishna Naik, Samrat Sen, **Ram Krishna Patra**, Manik Banik, and Mir Alimuddin;

Phys. Rev. A **106**, L040201 (2022) [**RKP 3**]

4. *Local quantum state marking;*

Samrat Sen, Edwin Peter Lobo, Sahil Gopalkrishna Naik, **Ram Krishna Patra**, Tathagata Gupta, Subhendu B. Ghosh, Sutapa Saha, Mir Alimuddin, Tamal Guha, Some Sankar Bhattacharya, and Manik Banik;

Phys. Rev. A **105**, 032407 (2022). [**RKP 4**]

5. *Timelike correlations and quantum tensor product structure;*

Samrat Sen, Edwin Peter Lobo, **Ram Krishna Patra**, Sahil Gopalkrishna Naik, Anandamay Das Bhowmik, Mir Alimuddin, and Manik Banik;

Phys. Rev. A **106**, 062406 (2022). [**RKP 5**]

6. *Mutually unbiased balanced functions and generalized random access codes;*

Vaisakh M, **Ram Krishna Patra**, Mukta Janpandit, Samrat Sen, Manik Banik, and Anubhav Chaturvedi;

Phys. Rev. A **104**, 012420 (2021). [**RKP 6**]

7. *Composition of Multipartite Quantum Systems: Perspective from Timelike Paradigm;*

Sahil Gopalkrishna Naik, Edwin Peter Lobo, Samrat Sen, **Ram Krishna Patra**, Mir Alimuddin,

Tamal Guha, Some Sankar Bhattacharya, and Manik Banik;

Phys. Rev. Lett. **128**, 140401 (2022). [**RKP 7**]

8. *Advantage of Hardy's nonlocal correlation in reverse zero-error channel coding;*

   Mir Alimuddin, Ananya Chakraborty, Govind Lal Sidhardh, **Ram Krishna Patra**, Samrat Sen, Snehasish Roy Chowdhury, Sahil Gopalkrishna Naik, and Manik Banik;

   Phys. Rev. A **108**, 052430 (2023). [**RKP 8**]

* Thesis related works

List of Communicated Work

1. *Overcoming Traditional No-Go Theorems: Quantum Advantage in Multiple Access Channels;*
   Ananya Chakraborty, Sahil Gopalkrishna Naik, Edwin Peter Lobo, **Ram Krishna Patra**, Samrat Sen, Mir Alimuddin, Amit Mukherjee, Manik Banik;
   arXiv: 2309.17263 [**RKP 9**]

2. *Asymptotic Birkhoff-Violation in Operational Theories: Thermodynamic Implications and Information Processing;*
   Ananya Chakraborty, Sahil Gopalkrishna Naik,Samrat Sen, **Ram Krishna Patra**, Pratik Ghosal, Mir Alimuddin, Manik Banik;
   arXiv: 2406.08803 [**RKP 10**]

3. *Local-Data-Hiding and Causal Inseparability: Probing Indefinite Causal Structures with Cryptographic Primitives;*
   Sahil Gopalkrishna Naik, Samrat Sen, **Ram Krishna Patra**, Ananya Chakraborty, Mir Alimuddin, Manik Banik, Pratik Ghosal;
   arXiv: 2407.20543*v*1 [**RKP 11**]

4. *Nonlocal Locking of Observable Quantities: A Faithful Signature of Nonclassical Correlations;*
   Mir Alimuddin, Snehasish Roy Chowdhury, **Ram Krishna Patra**, Subhendu B Ghosh, Tommaso Tufarelli, Gerardo Adesso, Manik Banik;
   arXiv:2407.08292 [**RKP 12**]

5. *Scalable & Noise-Robust Communication Advantage of Multipartite Quantum Entanglement;*
   Ananya Chakraborty, **Ram Krishna Patra**, Kunika Agarwal, Samrat Sen, Pratik Ghosal, Sahil Gopalkrishna Naik, Manik Banik;
   arXiv:2409.13223 [**RKP 13**]

1 Introduction

In the era of the second quantum revolution [1], quantum resources are poised to play a crucial role in communication. In the seminal superdense coding protocol, Bennett et al. demonstrated that quantum entanglement can double the classical capacity of a single qubit [2]. It has also been shown that quantum resources can reduce the complexity of classical communication in various tasks [3].

In this work, we aim to explore the role of different resources, such as shared randomness and Beyond Quantum States (BQS), in communication tasks. To harness these advantages in experimental settings, it is essential to certify the beneficial quantum resources. We have investigated the certification of Beyond Quantum States in comparison to quantum states.

Furthermore, communication tasks often have deeper foundational implications, such as in Bell games [4], Random Access Codes (RACs) [5, 6], and Information Causality [7]. We have also explored the foundational aspects of different communication tasks and the resources that prove useful in these scenarios.

2 A brief overview of the topic

2.1 Entanglement

Any physical system S is associated with complex inner product space known as Hilbert space ($\mathcal{H}_s$) and the state of a system is completely described by a density operator $\rho \in \mathcal{D}(\mathcal{H}_s)$, a positive Hermitian operator on the Hilbert space $\mathcal{H}_s$ with $\mathrm{Tr}(\rho) = 1$. Entanglement is a phenomenon of multiple particles, let us consider the simplest case where the number of component systems is two, *i.e.*, consider a composite quantum system AB with only two subsystems A and B. For such a bipartite system, any separable state ρ_{AB}^{sep} is of the following form :

$$\rho_{AB}^{sep} = \sum_j p_j \rho_A^j \otimes \rho_B^j \tag{1}$$

where $p_j \geq 0 \ \forall j$ and $\sum_j p_j = 1$. The set of all separable states forms a strict subset of the composite quantum state space $\mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Therefore, if $sep(\mathcal{H}_A \otimes \mathcal{H}_B)$ denotes the set of separable states, then $sep(\mathcal{H}_A \otimes \mathcal{H}_B) \subset \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Composite states that do not belong to $sep(\mathcal{H}_A \otimes \mathcal{H}_B)$ are called entangled states [8]. That is, if $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is entangled, then it cannot be expressed as a convex combination of product states *i.e.*, as in the form of Eq.(1). The necessary and sufficient criteria for the detection of entanglement for bipartite state $\rho_{AB} \in \mathcal{D}(\mathrm{C}^2 \otimes \mathrm{C}^2)$ and $\rho_{AB} \in \mathcal{D}(\mathrm{C}^2 \otimes \mathrm{C}^3)$ is given by Peres-Horodechki positive partial transpose (PPT) condition [9, 10].

2.2 Nonlocality

In the most general Bell theorem, two particles are with two parties separated in distant labs (say Alice and Bob) and they receive some random classical variable x and y respectively. They perform some measurements on their particle based on the classical input x and y yielding the output a and b respectively. Bell showed that under Locality and reality assumption the joint probability factorize [11] as follows

$$P(ab|xy) = \sum_{\lambda} P(\lambda)P(a|x,\lambda)P(b|y,\lambda). \tag{2}$$

In the special case of Bell-CHSH inequality, each party has two measurement choices with dichotomic outcome *i.e.* $x, y \in \{0, 1\}$ and $a, b \in \{-1, +1\}$ [12]. The Bell-CHSH inequality is given by

$$\mathcal{B}_{CHSH} = |\langle A_0 B_0\rangle + \langle A_0 B_1\rangle + \langle A_1 B_0\rangle - \langle A_1 B_1\rangle|$$

where $\langle A_i B_j\rangle = \sum_{a,b} abP(ab|x = i, y = j)$ is expectation value of the measurements. It is easy to check that for any local-Realistic theory

$$\mathcal{B}_{CHSH} \leq 2.$$

Quantum mechanics violates this bound *i.e.* there exists quantum state and measurements on Alice and Bob's side which will violate the above inequality. For instance, consider the state $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B)$ shared between Alice and Bob and take $A_0 = \sigma_z,\ A_1 = \sigma_x,\ B_0 = \frac{\sigma_z+\sigma_x}{\sqrt{2}}$ and $B_1 = \frac{\sigma_z-\sigma_x}{\sqrt{2}}$. Using Born rule the value of the Bell-CHSH expression becomes $2\sqrt{2}$. Later Cirelson proved that the maximal violation of Bell-CHSH in quantum is $2\sqrt{2}$ [13]. Popescu-Rorlich proves that there exists no-signaling correlation that violates the CHSH inequality up to algebraic maximum *i.e.* $\mathcal{B}_{CHSH}(NS) = 4$ [14]. Bell's theorem is a profound discovery with a deeper implication on the philosophy of nature.

2.3 Resources and Scenario of Communication Task

In a most general communication task, either Alice, Bob, or both receive some input, and either one or both provide some output. The referee will declare the payoff for all possible input-output statistics before the game starts. Alice and Bob's aim is to optimize the payoff collectively. Alice and Bob can utilize many resources to maximize their Payoff. These resources can be broadly divided into two categories: the common past resources and the direct communication resources.

(1.) **Common Past Resources (Type 1)**: These resources are created in the common past of Alice and Bob. They can be used to increase the payoff of different communication tasks even though these resources cannot be used to send any information. There are many examples of common past resources some of which are frequently used in the literature *e.g.* Classical Shared Randomness, Quantum Entanglement, No-signaling Correlation.

(2.) **Direct Communication Resources (Type 2)**: Alice takes some physical system and encodes her message in different states of the system and send to Bob, who may perform some measurements to obtain an outcome and optimize the required payoff accordingly. Depending on the type of the physical system the direct communication resources can be divided into the following three categories *e.g.* Classical system (cbit), Quantum system (qubit), Polygon System (gbit).

In any communication task, different parties receive inputs from the referee, and, as a result, some or all of them have to provide an outcome. The communication scenario varies depending on which party is receiving input and providing output. These scenarios can be divided into three classes

(1.) **H-FW Scenario** This is the most basic scenario involving two-party communication. Suppose, Alice and Bob are two distant parties. A Referee provides Alice some classical random variable $x \in \mathcal{X}$, while Bob needs to generate some classical random variable $b \in \mathcal{B}$ and return it to the Referee, according to the correlation $P(b|x)$ generated by Alice and Bob referee gives them some payoff $\beta : P(\mathcal{B}|\mathcal{X}) \to \mathbb{R}$ [15, 16]. The payoff can be defined on a set of correlations satisfying some constraints or on the achievability of some particular correlation. In this H-FW communication scenario, the celebrated no-go result by Holevo limits the information capacity of an n-level quantum system by the optimal value achievable with an n-level classical system[15].

(2.) **W-ANTV Scenario** This scenario is more involved and also distinct from the earlier scenario. In this case, the Referee provides random variables $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ to Alice and Bob, respectively, and only Bob needs to return an output random variable $b \in \mathcal{B}$. Accordingly, they will be given some payoff depending on the correlation generated by them $\beta : P(\mathcal{B}|\mathcal{X}, \mathcal{Y}) \to \mathbb{R}$. The famous no-go theorem of Holevo [15] later generalized by Frenkel and Weiner [16] is no longer applicable in this scenario. An N-level quantum system can be advantageous over an N-level classical system in this W-ANTV scenario which can be shown in the Random Access Coding (RAC) task[5, 6].

(3.) **Bell Scenario** As discussed earlier it is the most fundamental scenario and the prominent Bell theorem discarded the local-realistic model of nature in this scenario. In this case, both Alice and Bob are given classical random variables $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, respectively; and both of them

have to return some random variable $a \in \mathcal{A}$ and $b \in \mathcal{B}$, respectively to the referee[11, 12]. Accordingly, they are given some payoff $\beta : P(\mathcal{A}, \mathcal{B}|\mathcal{X}, \mathcal{Y}) \to \mathbb{R}$. Unlike the earlier two scenarios, they are not allowed to communicate with each other, *i.e.* resources of **Type-**2 are prohibited in this case. However, they can share some resources of **Type-**1.

2.4 Generalized Probability Theory

A GPT is specified by a list of system types and the composition rules specifying a combination of several systems. In the prepare and measure scenario, a system ($\mathcal{S}$) is described by the tuple (Ω, E). Here Ω represents the collection of normalized states, generally a compact-convex set embedded in some positive cone V_+ (collection of unnormalized states) of some real vector space V. On the other hand, E denotes the collection of effects, where an effect $e \in E$ corresponds to a linear functional $e : \Omega \mapsto [0, 1]$, with $e(\omega)$ denoting the success probability of filtering the effect e on the state $\omega \in \Omega$ when some measurement $M := \{e_i \mid e_i \in E \ \forall \ i, \ \sum_i e_i = u, \ \& \ u(\omega) = 1 \ \forall \ \omega \in \Omega\}$ is performed [17]. The unnormalized effects form a cone $V_+^\star$ which is dual to the state cone V_+. For instance, state cone of a quantum system associated with Hilbert space $\mathcal{H}$ consists of the set of positive operators $\mathcal{T}_+(\mathcal{H}) \subset \mathcal{T}(\mathcal{H})$ acting on $\mathcal{H}$, where $\mathcal{T}(\mathcal{H})$ denotes the set of all Hermitian operators on $\mathcal{H}$. A normalized state ρ is an element of $\mathcal{T}_+(\mathcal{H})$ with trace one and their collection is the convex-compact set of density operators $\mathcal{D}(\mathcal{H})$. A generic quantum measurement corresponds to positive operator valued measure (POVM) $M := \{\pi_i \mid \pi_i \in \mathcal{T}_+(\mathcal{H}), \ \sum_i \pi_i = \mathbf{1}_\mathcal{H}\}$, with $\mathbf{1}_\mathcal{H}$ being the identity operator on $\mathcal{H}$.

3 A brief summary of my work

3.1 Classical analog of quantum superdense coding and communication advantage of a single quantum system [RKP 1]

In the simplest communication scenario, a sender (Alice) transmits a message to a receiver (Bob) using physical systems. The quantum superdense coding protocol demonstrates how preshared quantum entanglement can double the classical communication capacity of a qubit channel [2]. However, according to Holevo's theorem, without preshared entanglement, the communication capacity of a quantum system is the same as that of a classical one [15, 16].

A key observation in this work is that while both quantum entanglement and classical shared randomness alone have no communication utility, classical shared randomness is often considered a free resource, unlike quantum entanglement. This assumption is challenged here, as classical communication is required to establish shared randomness, making it a costly resource. The utility of classical shared randomness has been explored in several fields, including quantum information theory [18, 19, 20, 21].

Let us consider a two-party task where Charlie is the manager of a chain of n Restaurants. On each day one of the Restaurants remains closed and which one is to be closed is decided by Charlie randomly with uniform probability. Charlie informs Alice which Restaurant is closed for the day. Another fellow, Bob, must visit one of Charlie's Restaurants for lunch each day. However, Bob does not know which Restaurant is closed for the day. He relies on Alice to communicate that information to him. Alice is restricted to communicating only 1-bit of information to Bob. The collective aim of Alice and Bob is to ensure the satisfaction of the two conditions: (1) Bob never visits a closed Restaurant and (2) Bob visits the k^{th} Restaurant with probability $\gamma_k \geq 0,\ \forall\ k \in \{1, \cdots, n\}$, where each γ_k can in general be different. We show that only a measure zero subclass of these games can be won perfectly if 1-bit of classical communication is allowed from the sender (Alice) to the receiver (Bob). However, all such games can be won deterministically if the classical communication line is assisted by additional shared randomness leading to the following theorem.

Theorem 1 *Shared randomness can increase the utility of a perfect 1 cbit classical channel in H-FW scenario.*

Following the quantum superdense coding phenomenon, wherein the single-shot communication utility of a perfect quantum channel is enhanced by the assistance of quantum correlation (entanglement), we call this the classical analog of superdense coding.

We have shown that for a class of the above-mentioned task quantum communication (qubit) is

advantageous over classical communication (cbit) in the absence of shared randomness. This does not contradict the Holevo theorem as even though communication capacity is bounded quantum communication can have enhanced capacity in terms of other utility.

Theorem 2 *In the absence of any preshared correlation utility of 1 qubit communication is higher than 1 bit of classical communication in H-FW scenario.*

Exploring the quantum advantage over classical communication we have given the following proposition where we have show the simultaneous use of quantum interference at the encoding step by Alice as well as at the decoding step by Bob is necessary for this particular advantage.

Proposition 1 *A qudit is no better than a c-dit, in H-FW scenario, if Alice encodes the inputs in d-dimensonal commuting density operators.*

Proposition 2 *An advantage of a qudit over its classical counterpart in the H-FW scenario requires the non-commutativity of the measurement operators at the receiver's end for decoding.*

We analyze the communication utility of a class of toy models, known as the polygon models [22]. Like two-level classical (bit) and quantum (qubit) systems, all these models allow at most two perfectly distinguishable states. However, with the help of our proposed game, we show that the communication utility of all the even-gons is strictly greater than the two-level classical system. We then propose a stricter version of the aforesaid game which we denote as $\mathbb{H}^n(1/(n-1))$. We prove that the $\mathbb{H}^4(1/3)$ game cannot be won perfectly with 1-bit of classical communication even when assisted with 1-bit of shared randomness.

Theorem 3 *In the H-FW scenario, the communication utility of 1 cbit channel is increased as the amount of assisting shared randomness is increased from 1 bit to* $\log 3$ *bit.*

More shared randomness is required to achieve the goal. This provides further justification to consider shared randomness as a costly resource. We have also demonstrated that qubit communication will result in perfect success for this task, leading to the following theorem.

Theorem 4 *There exists a task in the H-FW scenario where the utility of 1 qubit is higher than the utility of 1 cbit with the assistance of 1 bit of shared randomness i.e.* $C + 1SR \prec_{inst} Q$.

The $\mathbb{H}^4(1/3)$ task is an example where 1-bit Shared Randomness is not sufficient $\log 3$ bit Shared Randomness is required when the direct communication is restricted to 1 cbit. But in the case of quantum communication, no shared randomness is required, at this point, one might ask whether a similar kind of advantage is there in polygon theory. We have proven that the $\mathbb{H}^4(1/3)$ game cannot be won perfectly with a polygon bit.

Theorem 5 *There exist a task in the H-FW scenario where in the absence of any preshared correlation the utility of 1 qubit is greater than 1 polygon bit i.e. Polygon*$\prec_{inst}$ *Q.*

This work highlights the efficacy of classical shared randomness in enhancing the utility of classical communication. While much research has focused on leveraging common-past resources for direct communication, most studies have centered on non-classical resources [2, 23]. Classical shared randomness has also been explored in the Reverse Shannon Theorem for both classical and quantum scenarios [24, 25, 26, 27]. Notably, Cubitt's work [26] demonstrates that classical shared randomness can provide advantages, but quantum entanglement offers further improvements, establishing a hierarchy of shared correlations. However, this analysis does not limit the amount of shared correlation.

Our results build on this by considering a finite amount of shared randomness, leading to insights into the comparison between different levels of the same type of shared correlation. For example, perfect winning of our proposed game requires more shared randomness than 1 cbit + 1 bit of shared randomness (SR), while $\log 3$ bits of SR with 1 cbit suffice. This advantage resembles the quantum superdense coding phenomenon but uses classical resources only. Unlike superdense coding, where channel utility is quantified by mutual information, in our case, it is measured by the success probability in the Restaurant game we introduced.

The quantum advantage persists even under experimental noise, opening doors for future experiments with current quantum technologies. Our work also touches on recent advances in quantum foundations, such as Pusey-Barrett-Rudolph's result [28] and explores whether quantum advantages observed here connect with the no-go results like the one by Spekkens [29]. Additionally, we propose that our game could help certify communication resources, offering a practical way for Alice and Bob to verify the presence of direct communication using the $\mathbb{H}^n(1/n)$ game.

Our research opens new avenues, including the potential to explore whether classical shared randomness can empower other communication resources and uncover trade-offs between classical communication and shared randomness.

3.2 Principle of information causality rationalizes quantum composition [RKP 2]

Quantum mechanics is widely regarded as the most effective theory for describing nearly all natural phenomena. However, since its inception, the theory has sparked significant debates regarding its interpretation [30, 31, 32, 33], which continue to this day [34, 35]. The formalism of quantum mechanics begins with an abstract mathematical framework, specifically the Hilbert space, which calls for physical justification. The well-known no-signaling (NS) principle, which forbids instantaneous communication between distant parties, is insufficient by itself. It permits a wide range of mathematical models as

potential candidates for the theory of nature. Interestingly, in recent years, driven by advancements in quantum information theory, several new principles have been proposed to address the limitations of the NS principle [36, 37, 7, 38, 39, 40]. These principles effectively identify certain beyond-quantum NS correlations as unphysical, thereby providing physical justifications for quantum correlations.

In this work, we have focused on one such intriguing principle, known as information causality (IC), which was introduced nearly a decade ago [7]. IC can be seen as a generalization of the NS condition. According to the IC principle, Bob's information gain is limited to a maximum of m bits when m classical bits are communicated by Alice to him, even if Bob uses any of his local resources that may be correlated with Alice. Interestingly, several NS correlations violate this principle and are therefore considered unphysical [41, 42]. In essence, by restricting Direct communication resources (type-(II)) to classical, the IC principle deems some Common past resources (type-(I)) unphysical.

In this work, we have examined the reverse scenario—restricting type-(I) resources to classical. We have demonstrated that some type-(II) resources are incompatible with the IC principle and are thus unphysical. Specifically, we have considered a scenario where individual systems are quantum, but their composition can be modeled by any theory that satisfies the NS condition. Even for two quantum systems, multiple consistent compositions are possible, with quantum theory being just one example. The state space of the resulting system falls between two extremes: the maximal tensor product state space and the minimal tensor product state space [43]. While the maximal tensor product allows for exotic joint states that are not permitted in quantum theory, the minimal tensor product allows only separable states. Remarkably, the system obtained via the maximal tensor product of two elementary quantum systems violates the IC principle. This is significant because all NS correlations derived from beyond-quantum states are quantum-simulable, meaning they cannot produce beyond-quantum nonlocal correlations [44]. Furthermore, we have shown that even the minimal tensor product composition of two elementary quantum systems violates the IC principle. This result is especially striking, as the resulting theory is local by construction.

IC discards extreme compositions of elementary quantum.– We consider the scenario where Alice can communicate some abstract physical system to Bob with whom she can share pre-shared randomness. Within the mathematical framework of generalized probability theory (GPT) [17, 45, 46, 47, 48] such an elementary system $\mathcal{S}$ can be specified by the tuple of normalized state and effect spaces, *i.e.* $\mathcal{S} \equiv (\Omega, \mathcal{E})$. Sometime it is convenient to deal with unnormalized states and effects that form convex cones embedded in some $\mathbb{R}^n$. A GPT also captures the description of the composite system $\mathcal{S}^{AB} \equiv (\Omega^{AB}, \mathcal{E}^{AB})$ consisting of component subsystems $\mathcal{S}^A \equiv (\Omega^A, \mathcal{E}^A)$ and $\mathcal{S}^B \equiv (\Omega^B, \mathcal{E}^B)$. Under the restriction of NS and local tomography [49] the composite state space Ω^{AB} lies in between

two extremes – (i) the maximal tensor product state space and (ii) the minimal tensor product state space [43]. For instance, the state cone of a quantum system associated with a Hilbert space $\mathcal{H}$ is the set of positive semidefinite operators $\mathcal{P}(\mathcal{H}) \subset \mathcal{L}(\mathcal{H})$ acting on $\mathcal{H}$, whereas the normalized states are the set of density operators $\mathcal{D}(\mathcal{H})$; Here $\mathcal{L}(\mathcal{H})$ denotes the set of all linear operators acting on $\mathcal{H}$. For two quantum systems associated with Hilbert spaces $\mathcal{H}^A$ and $\mathcal{H}^B$ respectively, the state cone for maximal tensor product system is given by

$$\Omega_+^{AB}[\max] := \{\mathcal{W} \in \mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B) | \operatorname{tr}[\mathcal{W}(\pi_A \otimes \pi_B)] \geq 0 \ \forall \pi_A \in \mathcal{P}(\mathcal{H}^A) \text{ and } \ \forall \pi_B \in \mathcal{P}(\mathcal{H}^B)\}.$$

Clearly, $\Omega_+^{AB}[\max]$ contains all the quantum states $\mathcal{P}(\mathcal{H}^A \otimes \mathcal{H}^B)$, and furthermore it encompasses states that are not allowed in quantum theory. For example, entanglement witness that are not bona-fide quantum states [8] are valid states in this composite model. The effect cone is constructed in accordance with the no-restriction hypothesis that allows all mathematically consistent effects in the theory:

$$\mathcal{E}_+^{AB}[\max] := \{\pi \mid \pi = \sum_i \pi_i^A \otimes \pi_i^B; \ \pi_i^A \in \mathcal{P}(\mathcal{H}^A) \ \ \pi_i^B \in \mathcal{P}(\mathcal{H}^B)\}.$$

$\mathcal{E}_+^{AB}[\max]$ also forms a cone which is dual to the state cone $\Omega_+^{AB}[\max]$. In the other extreme, minimal tensor product contains only separable states, but the effect space gets enlarged here. More particularly, the role of state and effect cones of maximal tensor product are interchanged in the minimal case,

$$\Omega_+^{AB}[\min] := \mathcal{E}_+^{AB}[\max] \ \& \ \mathcal{E}_+^{AB}[\min] := \Omega_+^{AB}[\max].$$

Quantum composition lies in between and the state and effect cones becomes self dual in this case, *i.e.*, $\Omega_+^{AB}[Q] = \mathcal{P}(\mathcal{H}^A \otimes \mathcal{H}^B) = \mathcal{E}_+^{AB}[Q]$. Contributions of the present work are the following

Proposition 3 *The information capacity of the minimal composition of k local quantum systems, each of dimension d, is d^k, i.e., $\mathcal{I}\left((\mathbb{C}^d)^{\otimes_{\min} k}\right) = d^k$.*

Proposition 4 *The information capacity of the maximal composition of k local quantum systems, each of dimension d, is d^k, i.e., $\mathcal{I}\left((\mathbb{C}^d)^{\otimes_{\max} k}\right) = d^k$.*

Theorem 6 *Maximal tensor product of two elementary quantum violates the IC principle.*

Theorem 7 *Minimal tensor product of two elementary quantum violates the IC principle.*

The notion of composition is one of the guiding tools to fabricate our worldview – while complex objects are composed of elementary parts, some compositions are deemed implausible [50]. The idea becomes important even while constructing theories in Physics [51, 52]. In this work, we have studied this particular aspect while considering multiple quantum systems. Interestingly, we have shown that the principle of Information Causality [7] plays a crucial role in selecting the quantum composition among different mathematical possibilities. In the process it discards even a local theory as unphysical, which might make IC champion over the other principles [38, 39, 40]. As IC can derive some structural aspect of quantum theory it thus brings some physical ground justifying Hilbert space formulation of the theory. The potentiality arises from the communication aspect of the IC principle which invokes preparations (for encoding) and measurements (for decoding) of the involved systems and thus becomes more structure sensitive. As for future it would be quite interesting to see what other structural aspects of multipartite quantum systems can be rationalized with IC as the study in the present work is limited to bipartite compositions only.

3.3 Certifying beyond quantumness of locally quantum no-signalling theories through quantum input Bell test [RKP 3]

Correlations established through violations of Bell-type inequalities confirm the nonlocal nature of the physical world, as first proposed by Bell [11, 53, 54, 4]. Schrödinger emphasized the uniqueness of nonseparable quantum states, describing them as "...the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought" [55]. Over the years, quantum information science has leveraged these nonlocal correlations in several device-independent protocols, such as cryptographic key distribution [56], randomness certification [57], and dimension witnessing [58]. However, as shown by Cirel'son, the nonlocal strength of quantum correlations is fundamentally limited compared to general no-signaling (NS) correlations, as illustrated in the famous violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality [13, 59, 12].

Recent work by Barnum and colleagues provided a significant insight into the structure of bipartite correlations. They showed that the set of correlations attainable from POPT (positive on pure tensors) states is exactly the set of quantum correlations, concluding that if any experiment yielded nonlocal correlations beyond quantum mechanics, this would invalidate quantum theory even locally [44]. This result raises deep questions about the limits of quantum theory, particularly in relation to other possible physical theories that might go beyond quantum mechanics.

In this work, we have explored multipartite POPT state correlations in the generalized Bell scenario, introduced by Buscemi [60]. In contrast to the traditional Bell scenario—where distant parties

receive classical inputs—Buscemi's scenario allows them to receive quantum inputs. This generalization provides a new lens for studying nonlocality and enables the identification of quantum correlations that were previously hidden. Our results show that not all correlations obtained from bipartite POPT states are simulable within quantum mechanics. Specifically, some beyond-quantum POPT states generate correlations that cannot be reproduced by quantum states in certain quantum input games. Moreover, we have demonstrated that some POPT states produce classical-input-classical-output correlations that are not just quantum-simulable but also simulable classically, highlighting the limitations of the standard Bell scenario in capturing the full scope of beyond-quantum correlations.

Proposition 5 *There exist beyond quantum bipartite states yielding correlations that are classically simulable in the Standard Bell Scenario.*

Our Proposition 5 depicts that the correlation generated from this class of beyond quantum states (BQS) will always be Bell local in the standard classical input Bell scenario. This depicts the limitation of the standard Bell scenario and highlights the difficulty of certifying beyond quantum states (BQS) in this setup. This leads us to explore the BQS in the quantum input Bell scenario where the inputs can be quantum instead of classical. We have shown that the correlations generated by BQS in the quantum input Bell scenario are strictly larger than the correlation generated by quantum states. This phenomena paves the way to formulate a task of certifying the BQS in a semi-device-independent way.

Theorem 8 *For every beyond quantum state $W_{AB} \in \mathcal{W}(\mathcal{H}_A \otimes \mathcal{H}_B)$ there exists a semiquantum game $\mathbb{G}_{sq}$ such that $\mathcal{I}_{\mathbb{G}_{sq}}(W_{AB}) < 0$, while $\mathcal{I}_{\mathbb{G}_{sq}}(\rho_{AB}) \geq 0,\ \forall\ \rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$.*

Theorem 8 shows that in quantum input Bell scenario the set correlation generated by bipartite BQS ($\mathcal{GL}_{sq}$) is strictly larger than those generated by bipartite quantum state ($\mathcal{Q}_{sq}$). Acin et.al. [61] have already shown that in the case of a multipartite state with $N \geq 3$ barnum et.al. [44] result no longer hold *i.e.* $\mathcal{GL} \neq \mathcal{Q}$. They have characterized the BQS set for which the standard Bell correlations are quantum simulable. We are showing that in the quantum input Bell scenario, all BQS can generate a correlation that is not quantum simulable.

Theorem 9 *For every BQS $W_{A_1\cdots A_N} \in \mathcal{W}(\bigotimes_{i=1}^{N} \mathcal{H}_{A_i})$ there exists a semiquantum game $\mathbb{G}_{sq}$ such that $\mathcal{I}_{\mathbb{G}_{sq}}(W_{A_1\cdots A_N}) < 0$, whereas $\mathcal{I}_{\mathbb{G}_{sq}}(\rho_{A_1\cdots A_N}) \geq 0,\ \forall \rho_{A_1\cdots A_N} \in \mathcal{D}(\bigotimes_{i=1}^{N} \mathcal{H}_{A_i})$.*

Barnum et.al. have explored the limitation of the standard Bell scenario to reveal the beyond quantumness of POPT states. We have shown this can be captured in the quantum input Bell scenario and given a task (witness) corresponding to every POPT state. This gives us a way to certify beyond-quantumness of locally quantum no-signaling theories and in turn, suggests requirement of additional

information principles along with local quantum structure and no-signaling principle to isolate quantum correlations. More importantly, our work establishes that the additional principle(s) must be sensitive to the quantum signature of local inputs.

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